Journal of Statistical Physics, Vol. 44, Nos. 3/4, 1986

The Asymmetric Contact Process

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Received December 6, 1985; final March 25, 1986

We study a generalization of the Harris one-dimensional contact process in which the rates of infection to the right and left may be different.

KEY WORDS: Harris contact process; asymmetry; phase diagram; complete convergence theorem.

1. INTRODUCTION

Contact processes were first studied by Harris.(8) They can be considered as very idealized models for the spread of an infection and are also closely related to oriented percolation.(2,7) For reviews see Refs. 6, 7, and 11.

The most extensively studied of these models is the basic contact process in one dimension (BCP). Informally one can describe it in the following way: individuals are located at all the sites of the lattice $\mathbb{Z}$ (one individual at each site) and each one can be either infected or healthy. The infected individuals recover at a constant rate, which can be chosen as 1; and the healthy individuals become infected at a rate which is proportional to the number of infected nearest neighbors.

In this paper we consider a family of processes which generalizes the BCP in the sense that the rates of infection to the right and left may be different. The state of the system is determined by the set of infected individuals; so for each $(\lambda_r, \lambda_l) \in \mathbb{R}_+ \times \mathbb{R}_+$ (the rates of infection to the right and left) and $\eta \subset \mathbb{Z}$ consider the process $(\xi^{\eta}_{\lambda_r, \lambda_l}(t),\ t \geqslant 0)$, taking values on the set $\mathscr{P}(\mathbb{Z})$ of the subsets of $\mathbb{Z}$, starting from η at time 0 ($\xi^{\eta}_{\lambda_r, \lambda_l}(0) = \eta$ a.s.) and evolving according to the following local rates of change:

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822/44/3-4-15

0022-4715/86/0800-0505$05.00/0

$$\eta \to \eta/\{x\} \qquad \text{with rate} \quad 1 \qquad \text{if} \quad x \in \eta$$

$$\eta \to \eta \cup \{x\} \qquad \text{with rate} \begin{cases} \lambda_r & \text{if} \quad x \notin \eta, x-1 \in \eta, x+1 \notin \eta \\ \lambda_l & \text{if} \quad x \notin \eta, x-1 \notin \eta, x+1 \in \eta \\ \lambda_r + \lambda_l & \text{if} \quad x \notin \eta, x-1 \in \eta, x+1 \in \eta \end{cases}$$

The symmetric case $\lambda_r = \lambda_l$ corresponds to the BCP. Another particular case which has been studied in some detail in the literature is the one-sided (one-dimensional) contact process (OSCP)[6]; it corresponds to the cases $\lambda_r = 0$, $\lambda_l > 0$ or $\lambda_r > 0$, $\lambda_l = 0$.

Our motivation for studying this generalized one-dimensional contact process is the fact that it shows a behavior which is much richer than that of the particular case studied yet. For any $(\lambda_r, \lambda_l) \in \mathbb{R}^2_+$, Theorem 3.13 of Chapter III of Ref. 11 (first proved in Ref. 10) applies, implying the existence of at most two extremal invariant probability measures. One of them is trivially $\delta_\varnothing$ (point mass on $\varnothing$) and the other is the weak limit of the law of $\xi^{\mathbb{Z}}_{\lambda_r, \lambda_l}(t)$ as $t \to \infty$, denoted $\nu_{\lambda_r, \lambda_l}$. The process is ergodic for the values of (λ_r, λ_l) such that $\nu_{\lambda_r, \lambda_l} = \delta_\varnothing$.

For the BCP there is a critical value $\lambda_c \in (0, \infty)$ such that if $\lambda_r = \lambda_l < \lambda_c$ the process is ergodic. If $\lambda_r = \lambda_l = \lambda > \lambda_c$, then $\nu_{\lambda,\lambda} \neq \delta_\varnothing$ and the so-called complete convergence theorem (CCT) holds[1]:

$$\forall \eta \subset \mathbb{Z}, \ \xi^\eta_{\lambda,\lambda}(t) \to \beta \nu_{\lambda,\lambda} + (1-\beta)\, \delta_\varnothing$$

where $\beta = P(\xi^\eta_{\lambda,\lambda}(t) \neq \varnothing, \ \forall t \geq 0)$. The behavior of the BCP when $\lambda_r = \lambda_l = \lambda_c$ is still an open problem; one does not even know in this case whether the process is ergodic or not.

For the OSCP there is also a critical value $\lambda_c^+ \in (0, \infty)$ such that the process is ergodic if $\min(\lambda_r, \lambda_l) = 0$, $\max(\lambda_r, \lambda_l) < \lambda_c^+$. If $\min(\lambda_r, \lambda_l) = 0$ and $\max(\lambda_r, \lambda_l) > \lambda_c^+$, then $\nu_{\lambda_r, \lambda_l} \neq \delta_\varnothing$, but the CCT does not hold; instead one has[6]:

$$\forall \eta \subset \mathbb{Z} \text{ s.t. } |\eta| < \infty, \ \xi^\eta_{\lambda_r, \lambda_l}(t) \to \delta_\varnothing \text{ where } |\eta| \text{ is the cardinality of } \eta \tag{1.1}$$

$$\text{There are configurations } \eta \subset \mathbb{Z} \text{ s.t. } \xi^\eta_{\lambda_r, \lambda_l}(t) \text{ does not converge in law} \tag{1.2}$$

The system that we consider in this paper shows a rich "phase diagram" (see Fig. 1). There is a region $\mathscr{A}$ in which it is ergodic; a region $\mathscr{B}_1$ in which it is not ergodic and (1.1) and (1.2) hold; and a region $\mathscr{B}_2$ where the system is not ergodic and the CCT holds. The more interesting behavior occurs, however, for (λ_r, λ_l) on the boundary between $\mathscr{B}_1$ and $\mathscr{B}_2$; in that case we prove that

$$\forall \eta \subset \mathbb{Z} \text{ s.t. } |\eta| < \infty$$

$$\xi^\eta_{\lambda_r, \lambda_l}(t) \to (\beta/2)\, \nu_{\lambda_r, \lambda_l} + (1 - \beta/2)\, \delta_\varnothing \tag{1.3}$$

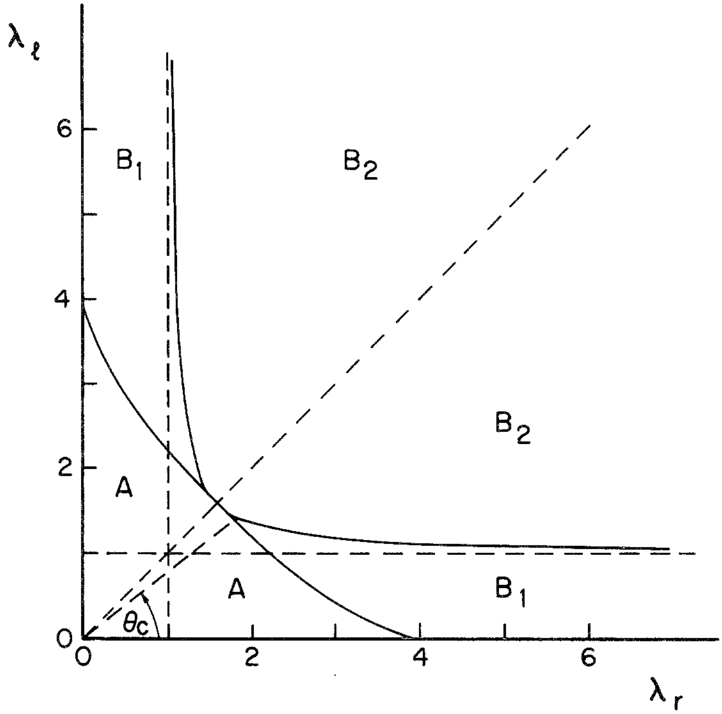

Fig. 1. The phase diagram. The author's conjectures were incorporated except for $\theta_c = \pi/4$.

where, as before, $\beta = P(\xi^{\eta}_{\lambda_r, \lambda_l}(t) \neq \emptyset, \forall t \geqslant 0)$. We prove also that there are configurations η with infinite cardinality such that the same is true (with $\beta = 1$) and in some cases we specify such configurations. And there are also configurations η such that $\xi^{\eta}_{\lambda_r, \lambda_l}(t)$ does not converge.

The behavior on the boundary between $\mathscr{A}$ and $\mathscr{B}_1 \cup \mathscr{B}_2$ is as complicated as on the critical point λ_c for the BCP and we have nothing to say about it.

Besides the behavior of the system for (λ_r, λ_l) on each of these regions we get results about the shape and size of the regions.

The techniques used for the BCP mostly generalize to the asymmetric case. We suppose that the reader is familiar with these techniques.

This paper is organized in the following way. In Sect. 2 we construct the processes using a directed percolation structure (DPS), introduce the basic notation, and recall the main properties.

In Sec. 3 we prove some of the simplest properties of the boundary l_1 of $\mathscr{A}$.

In Section 4 we prove some results about the "edge processes" $r_t = \max \xi^{\mathbb{Z}_-}_{\lambda_r, \lambda_l}(t)$ and $l_t = \min \xi^{\mathbb{Z}_+}_{\lambda_r, \lambda_l}(t)$, where $\mathbb{Z}_- = \{..., -2, -1, 0\}$, $\mathbb{Z}_+ =$

$\{0, 1, 2,...\}$. Some of these results are simple extensions of the similar statements in the symmetric case, but others are of interest only in the asymmetric case.

In Sec. 5 we introduce $\mathscr{B}_1, \mathscr{B}_2$ and the boundary l_2 of $\mathscr{B}_2$.

In Sec. 6 we verify that the construction introduced by Durrett and Griffeath[3] relating the BCP to a one-dependent percolation process can be generalized to the asymmetric case. We employ this trick and its consequences to get more information about the geometry of l_1 and l_2.

In Sec. 7 we employ the "invariant measure as viewed from the edge" in order to get more information about l_2. The existence of such a probability measure was proven by Durrett[2] for a process in discrete time (oriented percolation) closely related to the BCP. His results can be extended easily to the BCP and the asymmetric contact process. The uniqueness of this measure was proven, among other things, by Galves and Presutti.[4] We employ the techniques and results in Ref. 4 to get some results which are of interest even in the symmetric case (Theorem 18).

In Sec. 8 we prove (1.3) and related results when $(\lambda_r, \lambda_l) \in l_2$. The basic ingredients for the proof are the central limit theorem for the edge processes proved in Ref. 4 and the technique used by Griffeath to prove the lemma on page 383 of Ref. 5.

Finally, in Sec. 9 we list some open problems.

2. CONSTRUCTION OF THE PROCESS. NOTATION AND BASIC PROPERTIES

In order to prove some of our results we need the following construction of the contact process using a directed percolation structure (DPS) on $\mathbb{Z} \times \mathbb{R}_+$. We employ the notation in Ref. 4, which is very clear. For each $x \in \mathbb{Z}$, let $(U_n^{(x,x+1)}: n = 1, 2,...)$, $(U_n^{(x,x-1)}: n = 1, 2,...)$, and $(U_n^x: n = 1, 2,...)$ be three independent Poisson point processes in $\mathbb{R}_+$ with intensity λ_r, λ_l, and 1 respectively. We suppose that for x varying in $\mathbb{Z}$ these poisson point processes are all independent and we denote by (Ω, Σ, P) the probability space in which they are defined.

Given s, t in $\mathbb{R}_+$ with $s < t$, x and y in $\mathbb{Z}$, and ω in Ω we will say that there is a ω-path from (x, s) to (y, t), and write $(x, s)) \to^{\omega} (y, t)$ if there exists a finite sequence of points $x_0, x_1,..., x_k$ with $x_0 = x$, $x_k = y$ and $|x_i - x_{i+1}| = 1$ and integers $n_1,..., n_k$ such that $s < U_{n_1}^{(x_0,x_1)}(\omega) < \cdots < U_{n_k}^{(x_{k-1},x_k)}(\omega) < t$ and moreover that the following situations

$$U_{n_{j-1}}^{(x_{j-1},x_j)}(\omega) \leqslant U_m^{x_j}(\omega) \leqslant U_{n_{j+1}}^{(x_j,x_{j+1})}(\omega)$$

$$s \leqslant U_m^x(\omega) \leqslant U_{n_1}^{(x,x_1)}(\omega), \qquad U_{n_{k-1}}^{(x_{k-1},y)}(\omega) \leqslant U_m^y(\omega) \leqslant t$$

do not occur for any index j and m.

The contact process takes values in the set $\mathscr{P}(\mathbb{Z})$ of subsets of $\mathbb{Z}$, whose elements we will usually call a "configuration." Given a configuration η, two times $s<t$, and a point ω in Ω, we define the configuration

$$(\xi_{\lambda_r,\lambda_l}^{(\eta,s)}(t))(\omega)=\{x\in\mathbb{Z}\colon \exists y\in\eta \text{ s.t. } (y,s)\xrightarrow{\omega}(x,t)\}$$

$\xi_{\lambda_r,\lambda_l}^{(\eta,s)}(t)\colon \Omega\to\mathscr{P}(\mathbb{Z})$ is then a random configuration.

If $s=0$, we will write only $\xi_{\lambda_r,\lambda_l}^{\eta}(t)$ instead of $\xi_{\lambda_r,\lambda_l}^{(\eta,0)}(t)$. If $\eta=\{x\}$, $x\in\mathbb{Z}$, we will write only $\xi_{\lambda_r,\lambda_l}^{(x,s)}(t)$. If there is no possibility of confusion, we will omit λ_r, λ_l in the notation.

The process $(\xi_{\lambda_r,\lambda_l}^{\eta}(t),\ t\geqslant 0)$ constructed on (Ω,Σ,P) is a version of the contact process as defined in the introduction. This construction using a DPS is particularly suitable in order to couple processes starting from different configurations in a useful way (see P1 below).

One can enlarge the space (Ω,Σ,P) in order to define processes starting with a random initial configuration independent of the DPS. In this case we will denote the process by $(\xi_{\lambda_r,\lambda_l}^{\mu}(t),\ t\geqslant 0)$ where μ is the law of the initial configuration.

We will use the following notation:

$$\tau_{\lambda_r,\lambda_l}^{(\eta,s)}=\inf\{t\geqslant s\colon \xi_{\lambda_r,\lambda_l}^{(\eta,s)}(t)=\varnothing\}$$

$$\tau_{\lambda_r,\lambda_l}^{(\eta,0)}=\tau_{\lambda_r,\lambda_l}^{\eta}$$

$$\tau_{\lambda_r,\lambda_l}^{(\{x\},s)}=\tau_{\lambda_r,\lambda_l}^{(x,s)}$$

$$\tau_{\lambda_r,\lambda_l}^{(\eta,s)}=\tau^{(\eta,s)} \text{ if no confusion is possible.}$$

Given $\eta\subset\mathbb{Z}$, we denote the cardinality of η by $|\eta|$, and define also

$$r(\eta)=\sup\eta,\qquad l(\eta)=\inf(\eta)$$

Finally we write

$$r_{\lambda_r,\lambda_l}^{\eta}(t)=r(\xi_{\lambda_r,\lambda_l}^{\eta}(t)),\qquad l_{\lambda_r,\lambda_l}^{\eta}(t)=l(\xi_{\lambda_r,\lambda_l}^{\eta}(t))$$

and abbreviate $r_t^{\eta}=r_{\lambda_r,\lambda_l}^{\eta}(t)$, $l_t^{\eta}=l_{\lambda_r,\lambda_l}^{\eta}(t)$, when it is possible. For $\eta=\mathbb{Z}_-=\{...,-2,-1,0\}$ we write $r_t=r_{\lambda_r,\lambda_l}(t)=r_{\lambda_r,\lambda_l}^{\mathbb{Z}_-}(t)$, and for $\eta=\mathbb{Z}_+=\{0,1,2,...\}$ we write $l_t=l_{\lambda_r,\lambda_l}(t)=l_{\lambda_r,\lambda_l}^{\mathbb{Z}_+}(t)$.

Some basic properties that follow from the construction and which the reader familiar with the BCP can easily prove are the following:

(P1) Additivity: If $A,B,C\subset\mathbb{Z}$ and $A=B\cup C$, then

$$\xi^{(A,s)}(t)=\xi^{(B,s)}(t)\cup\xi^{(C,s)}(t) \text{ for any } t\geqslant s\geqslant 0.$$

(P2) Symmetric-duality: If $A, B \subset \mathbb{Z}$, then

$$P(\xi^A_{\lambda_r,\lambda_l}(t) \cap B = \varnothing) = P(\xi^B_{\lambda_l,\lambda_r}(t) \cap A = \varnothing)$$

(P3) On the event $[\tau^0 > t]$ the relation

$$\xi^0(t) \cap [l_t, r_t] = \xi^\eta(t) \cap [l_t, r_t]$$

holds for any η such that $0 \in \eta$.

(P4) The law of $\xi^{\mathbb{Z}}_{\lambda_r,\lambda_l}(t)$ converges weakly as $t \to \infty$. The limit is a translation-invariant measure which will be denoted by $\nu_{\lambda_r,\lambda_l}$. Its density will be denoted by $\rho(\lambda_r, \lambda_l) = \nu_{\lambda_r,\lambda_l}(\eta\colon 0 \in \eta)$.

(P5) $\rho(\lambda_r, \lambda_l) = \rho(\lambda_l, \lambda_r)$

(P6) $\rho(\lambda_r, \lambda_l) = P(\tau^0_{\lambda_r,\lambda_l} = \infty)$

(P7) If $\lambda'_r \geqslant \lambda_r$ and $\lambda'_l \geqslant \lambda_l$ then $\rho(\lambda'_r, \lambda'_l) \geqslant \rho(\lambda_r, \lambda_l)$.

(P8) If $\lambda'_r \geqslant \lambda_r$ and $\lambda'_l \geqslant \lambda_l$, one can construct

$$\{(\xi^\eta_{\lambda_r,\lambda_l}(t), t \geqslant 0)\colon \eta \subset \mathbb{Z}\} \quad \text{and} \quad \{(\xi^\eta_{\lambda_r,\lambda_l}(t), t \geqslant 0)\colon \eta \subset \mathbb{Z}\}$$

on the same probability space in such a way that for any $\eta \subset \mathbb{Z}$ and $t \geqslant 0$, $\xi^\eta_{\lambda_r,\lambda_l}(t) \subset \xi^\eta_{\lambda'_r,\lambda'_l}(t)$.

Since we will need this construction, we will specify it now. For each $x \in \mathbb{Z}$, take $(U^{(x,x+1)}_n\colon n = 1, 2,...)$, $(U^{(x,x-1)}_n\colon n = 1, 2,...)$, and $(U^x_n\colon n = 1, 2,...)$ as before. Take also other two-independent Poisson point processes $(V^{(x,x+1)}_n\colon n = 1, 2,...)$ and $(V^{(x,x-1)}_n\colon n = 1,2,...)$ with intensity $\lambda'_r - \lambda_r$ and $\lambda'_l - \lambda_l$. We suppose that for x varying in $\mathbb{Z}$ these Poisson point process are all independent. Define $(W^{(x,y)}_n\colon n = 1, 2,...)$ as the Poisson point process obtained by the superposition of $(U^{(x,y)}_n)$ and $(V^{(x,y)}_n)$, $y = x - 1$, $x + 1$. Finally construct $(\xi^\eta_{\lambda_r,\lambda_l}(t))$ as before and $(\xi^\eta_{\lambda'_r,\lambda'_l}(t))$ in a similar way, using $(W^{(x,y)}_n)$ instead of $(U^{(x,y)}_n)$, $y = x - 1$, $x + 1$, but using the same $(U^{(x)}_n)$. When working with this enlarged space we will denote it also by (Ω, Σ, P).

For several purposes it will be convenient to localize points (λ_r, λ_l) on the phase diagram using polar coordinates $\sigma \in \mathbb{R}_+$, $\theta \in [0, \pi/2]$ defined by

$$\lambda_r = \sigma \cos\theta \qquad \lambda_l = \sigma \sin\theta$$

But instead of σ we will in general use $\lambda = (\lambda_r + \lambda_l)/2 = \sigma(\sin\theta + \cos\theta)/2$. For convenience we define $c(\theta) = 2\cos\theta/(\sin\theta + \cos\theta)$, $s(\theta) = 2\sin\theta/(\cos\theta + \cos\theta)$. Then

$$\lambda_r = \lambda c(\theta) \qquad \lambda_l = \lambda s(\theta)$$

For each fixed θ the family of processes obtained by varying λ will play an important role. This families will be called "radial families," for an obvious reason. When $\theta=\pi/4$, we have the BCP; and when $\theta=0$ or $\theta=\pi/2$, the OSCP.

Warning about notation: We adopt the convention that C and γ will denote constants, but from equality to equality their values may change.

3. FIRST CRITICAL LINE

Define $\bar{\rho}(\lambda, \theta)=\rho(\lambda c(\theta), \lambda s(\theta))$. Then P7 in Sec. 2 implies that the function $\lambda \to \bar{\rho}(\lambda, \theta)$ in increasing for each θ. As for the BCP, define

$$\lambda_{c1}(\theta)=\sup\{\lambda \in \mathbb{R}_+ : \bar{\rho}(\lambda, \theta)=0\}$$

$$\sigma_{c1}(\theta)=2\lambda_{c1}(\theta)/(\sin\theta+\cos\theta)$$

It was demonstrated in Ref. 9 that if $\lambda>2$ the process is not ergodic and

$$\rho(\lambda_r, \lambda_l)=\bar{\rho}(\lambda, \theta)\geqslant\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{1}{2\lambda}}$$

On the other hand, for any finite $\eta\subset\mathbb{Z}$, $\operatorname{diam}(\xi^{\eta}_{\lambda_r,\lambda_l}(t))=r(\xi^{\eta}_{\lambda_r,\lambda_l}(t))-l(\xi^{\eta}_{\lambda_r,\lambda_l}(t))$ increases one unit at rate λ_r, $\lambda_l=2\lambda$ and decreases at least one unit at rate 2. So if $\lambda<1$, $\bar{\rho}(\lambda, \theta)=P(\tau^{\{0\}}=\infty)=0$. The process is then ergodic.

From the last two paragraphs we conclude that for any $\theta\in[0, \pi/2]$, $\lambda_{c1}(\theta)\in[1, 2]$.

Theorem 1. $\theta\to\lambda_{c1}(\theta)$ is a continuous function.

Proof. Consider a fixed $\theta\in[0, \pi/2]$. Given an $\varepsilon>0$ we take $\lambda_1=\lambda_{c1}(\theta)-\varepsilon/2$, $\lambda_2=\lambda_{c1}(\theta)-\varepsilon/4$, $\lambda_3=\lambda_{c1}(\theta)+\varepsilon/4$, $\lambda_4=\lambda_{c1}(\theta)+\varepsilon/2$. Take now $\delta'>0$ such that

$$\begin{aligned} 0<\delta<\delta' \Rightarrow \lambda_4(s(\theta)-\delta)&>\lambda_3 s(\theta)\\ \lambda_4(c(\theta)-\delta)&>\lambda_3 c(\theta)\\ \lambda_1(s((\theta)+\delta)&<\lambda_2 s(\theta)\\ \lambda_1(c(\theta)+\delta)&<\lambda_2 c(\theta)\end{aligned}$$

Now there exists a $\delta>0$ such that $|\theta'-\theta|<\delta\Rightarrow|s(\theta)-s(\theta')|<\delta'$ and $|c(\theta)-c(\theta')|<\delta'$. Then, if $|\theta-\theta'|<\delta$, it follows that:

(i) $\lambda_4 s(\theta')>\lambda_3 s(\theta)$ and $\lambda_4 c(\theta')>\lambda_3 c(\theta)$. Then, since $\lambda_3>\lambda_{c1}(\theta)$,

$$P(\tau^0_{\lambda_4 c(\theta'), \lambda_4 s(\theta')}=\infty)\geqslant P(\tau^0_{\lambda_3 c(\theta), \lambda_3 s(\theta)}=\infty)>0$$

Therefore $\lambda_4 \geqslant \lambda_{c1}(\theta')$.

(ii) $\lambda_1 s(\theta') < \lambda_2 s(\theta)$ and $\lambda_1 c(\theta') < \lambda_2 c(\theta)$. Then, since $\lambda_2 < \lambda_{c1}(\theta)$,

$$P(\tau^0_{\lambda_1 c(\theta'), \lambda_1 s(\theta')} = \infty) \leqslant P(\tau^0_{\lambda_2 c(\theta), \lambda_2 s(\theta)} = \infty) = 0$$

Therefore $\lambda_1 \leqslant \lambda_{c1}(\theta')$.

In conclusion $|\theta - \theta'| < \delta \Rightarrow \lambda_1 \leqslant \lambda_{c1}(\theta') \leqslant \lambda_4 \Rightarrow |\lambda_{c1}(\theta') - \lambda_{c1}(\theta)| < \varepsilon$. ∎

Remark. Since for each θ, σ is a continuous function of λ, $\theta \to \sigma_{c1}(\theta)$ is a continuous function and defines a curve on the phase diagram (λ_r, λ_l).

We divide the phase diagram (λ_r, λ_l) in the subcritical region $\mathscr{A}$, the supercritical region $\mathscr{B}$, and the curve l_1 which separates them, defined as follows:

$$\mathscr{A} = \{(\lambda_r, \lambda_l) \in \mathbb{R}_+ \times \mathbb{R}_+ : \lambda < \lambda_{c1}(\theta)\}$$

$$\mathscr{B} = \{(\lambda_r, \lambda_l) \in \mathbb{R}_+ \times \mathbb{R}_+ : \lambda > \lambda_{c1}(\theta)\}$$

$$l_1 = \{(\lambda_r, \lambda_l) \in \mathbb{R}_+ \times \mathbb{R}_+ : \lambda = \lambda_{c1}(\theta)\}$$

Theorem 2. $(\lambda_r, \lambda_l) \to \rho(\lambda_r, \lambda_l)$ is a continuous function on $\mathscr{B}$.

Proof. For $(\lambda_r, \lambda_l) \in \mathbb{R}^2_+$ define $B_\delta(\lambda_r, \lambda_l) = \{(x, y) \in \mathbb{R}^2_+ : |x - \lambda_r| < \delta, |y - \lambda_l| < \delta\}$.

First consider $(\lambda_r, \lambda_l) \in \mathscr{B}$ such that $\lambda_r, \lambda_l \neq 0$. Then P7 of Sec. 2 implies that for δ sufficiently small

$$\sup_{(\lambda'_r, \lambda'_l) \in B_\delta(\lambda_r, \lambda_l)} |\rho(\lambda'_r, \lambda'_l) - \rho(\lambda_r, \lambda_l)| \leqslant \rho(\lambda_r + \delta, \lambda_l + \delta) - \rho(\lambda_r - \delta, \lambda_l - \delta)$$

So it is enough to prove that $x \to \rho(\lambda_r + x, \lambda_l + x)$ is continuous at $x = 0$. Theorem 1 implies the existence of $\delta > 0$ such that $\{(\lambda_r + x, \lambda_l + x): x \in (-\delta, \delta)\} \subset B$. The continuity at $x = 0$ now follows from the same argument used in the proof of the analogous theorem for the BCP (Theorem 1.6. (d) of Chapter VI of Ref. 11).

For $(\lambda_r, \lambda_l) \in \mathscr{B}$ such that $\lambda_l = 0$

$$\sup_{(\lambda'_r, \lambda'_l) \in B_\delta(\lambda_r, \lambda_l)} |\rho(\lambda'_r, \lambda'_l) - \rho(\lambda_r, \lambda_l)| \leqslant \rho(\lambda_r + \delta, \lambda_l + \delta) - \rho(\lambda_r - \delta, \lambda_l)$$

Now the proof can be completed as before using the continuity of the functions $x \to \rho(\lambda_r + x, 0)$ and $y \to \rho(\lambda_r + y, y)$, $(y \geqslant 0)$, at $x = 0$ and $y = 0$.

The case $\lambda_r = 0$ is analogous. ∎

4. EDGE PROCESSES

Recall the definitions of r_t aand l_t in the introduction and define

$$\alpha_t^1(\lambda_r, \lambda_l) = E(r_t)$$

$$\alpha_t^2(\lambda_r, \lambda_l) = -E(l_t) = \alpha_t^1(\lambda_l, \lambda_r)$$

The following two theorems can be proven in complete analogy with the corresponding theorems for the BCP (Theorems 2.19 and 2.24 of Chapter VI of Ref. 11, first proven in Ref. 1).

Theorem 3.

(a) $\alpha_1 = \lim_{t\to\infty} \frac{\alpha_t^1}{t} = \inf_{t>0} \frac{\alpha_t^1}{t} \in [-\infty, \infty)$

$\alpha_2 = \lim_{t\to\infty} \frac{\alpha_t^2}{t} = \inf_{t>0} \frac{\alpha_t^2}{t} \in [-\infty, \infty)$

(b) $\lim_{t\to\infty} \frac{r_t}{t} = \alpha_1$ a.s.

$\lim_{t\to\infty} \frac{l_t}{t} = -\alpha_2$ a.s.

(c) If $\alpha_1 > -\infty$, then $\lim_{t\to\infty} E\left|\frac{r_t}{t} - \alpha_1\right| = 0$

If $\alpha_2 > -\infty$, then $\lim_{t\to\infty} E\left|\frac{l_t}{t} + \alpha_2\right| = 0$

Notation: When we need to specify the dependence on (λ_r, λ_l), we write $\alpha_i(\lambda_r, \lambda_l)$ $(i = 1, 2)$. Observe that $\alpha_2(\lambda_r, \lambda_l) = \alpha_1(\lambda_l, \lambda_r)$.

Theorem 4.

$$\alpha_1(\lambda_r + \delta, \lambda_l) \geq \alpha_1(\lambda_r, \lambda_l) + \delta$$

$$\alpha_2(\lambda_r, \lambda_l + \delta) \geq \alpha_2(\lambda_r, \lambda_l) + \delta$$

P8 in the introduction implies that $\alpha_1(\lambda_r, \lambda_l)$ is nondecreasing in both arguments. Theorem 4 shows that it is strictly increasing in λ_r. We will prove now that it is also strictly increasing in λ_l.

Theorem 5. For each fixed λ_r, $\lambda_l \to \alpha_1(\lambda_r, \lambda_l)$ is a strictly increasing function. For each fixed λ_l, $\lambda_r \to \alpha_2(\lambda_r, \lambda_l)$ is a strictly increasing function.

Proof. Both statements are, of course, analogous and we prove the first one. Take (λ_r, λ_l) and $(\lambda_r, \lambda_l'$ such that $\lambda_l' > \lambda_l$. Consider now the coupled versions of $(\xi^{\eta}_{\lambda_r, \lambda_l}(t),\ t \geqslant 0)$ and $(\xi^{\eta}_{\lambda_r, \lambda_l'}(t),\ t \geqslant 0)$ as constructed in Sec. 2 for the proof of P8 (here $\lambda_r' = \lambda_r$). Define, using the coupling, the following objects for $n \geqslant k$:

$$\eta_n^k = \xi^{(\zeta, k)}_{\lambda_r, \lambda_l'}(n), \qquad \text{where } \zeta = \xi^{\mathbb{Z}_-}_{\lambda_r, \lambda_l}(k)$$

Then

$$\begin{aligned} &E(r(\xi^{\mathbb{Z}_-}_{\lambda_r, \lambda_l'}(n))) - r(\xi^{\mathbb{Z}_-}_{\lambda_r, \lambda_l}(n)) \\ &\quad = E(r(\eta_n^0) - r(\eta_n^1)) + E(r(\eta_n^1) - r(\eta_n^2)) + \cdots \\ &\quad + E(r(\eta_n^{n-2}) - r(\eta_n^{n-1})) + E(r(\eta_n^{n-1}) - r(\eta_n^n)) \end{aligned} \tag{4.1}$$

The proof will be complete if we prove that each term on the r.h.s. of (4.1) is bounded below by the same strictly positive number p. In fact, define

$$\begin{aligned} p = P[&U_1^{(0,1)} > 1,\ U_1^{(0,-1)} > 1,\ U_1^{(-1)} < V_1^{(0,-1)} < U_1^{(0)} < 1, \\ &U_1^{(-2,-1)} > 1,\ U_2^{(-1)} > 1] \end{aligned}$$

This choice is such that on the event above, whose probability is p, the following occurs for any η such that $r(\eta) = 0$:

$$r(\xi^{\eta}_{\lambda_r, \lambda_l}(1)) \leqslant -2 \qquad \text{and} \qquad r(\xi^{\eta}_{\lambda_r, \lambda_l'}(1)) = -1$$

Using the relation (2.23) of Chapter VI of Ref. 11: if $B \subset (-\infty, -1]$, then $E(r_t^{B \cup \{0\}}) - E(r_t^B) \geqslant 1$ (valid also in the asymmetric case), and the properties of the Poisson processes, it follows that

$$E(r(\eta_n^k) - r(\eta_n^{k+1})) \geqslant P(r(\eta_n^k) > r(\eta_n^{k+1})) \geqslant p \qquad (0 \leqslant k \leqslant n-1) \tag{4.2}$$

Combining (4.1) and (4.2) yields

$$\alpha_n^1(\lambda_r, \lambda_l') - \alpha_n^1(\lambda_r, \lambda_r) \geqslant np$$

and the thesis follows by the definition of α_1, since $p > 0$. ∎

Definition.

$$\alpha(\lambda_r, \lambda_l) = \frac{\alpha_1(\lambda_r, \lambda_l) + \alpha_2(\lambda_r, \lambda_l)}{2} = \frac{\alpha_1(\lambda_r, \lambda_l) + \alpha_1(\lambda_l, \lambda_r)}{2}$$

Theorem 6.

(a) $\alpha(\lambda_r, \lambda_l) < 0 \Rightarrow \rho(\lambda_r, \lambda_l) = 0$
(b) $\alpha(\lambda_r, \lambda_l) > 0 \Rightarrow \rho(\lambda_r, \lambda_l) > 0$
(c) $(\lambda_r, \lambda_l) \in \mathscr{A} \Rightarrow \alpha(\lambda_r, \lambda_l) < 0$
(d) $(\lambda_r, \lambda_l) \in \mathscr{B} \Rightarrow \alpha(\lambda_r, \lambda_l) > 0$
(e) $\alpha(\lambda_r, \lambda_l) = 0 \Rightarrow (\lambda_r, \lambda_l) \in l_1$
(f) $(\lambda_r, \lambda_l) \in l_1 \Rightarrow \alpha(\lambda_r, \lambda_l) \geqslant 0$

Proof. Analogous to the proof of Theorem 2.27 of Chapter VI of Ref. 11. For the parts (c) to (f), use radial families.

So, if we define $\bar{\alpha}(\lambda, \theta) = \alpha(\lambda c(\theta), \lambda s(\theta))$, it follows that

$$\lambda_{c1}(\theta) = \sup\{\lambda \geqslant 0: \bar{\alpha}(\lambda, \theta) \leqslant 0\}$$

The statement (f) above will be strengthened in Section 6. The statement (c) can be improved as in Ref. 11 (Theorem 3.4 and Corollary 3.8 of Chapter 6 of Ref. 11 have analogues in the asymmetric case) to

$$(\lambda_r, \lambda_l) \in \mathscr{A} \Rightarrow \alpha_1(\lambda_r, \lambda_l) = \alpha_2(\lambda_r, \lambda_l) = -\infty$$

Note that therefore

$$\alpha_1(\lambda_r, \lambda_l) = -\infty \Leftrightarrow \alpha_2(\lambda_r, \lambda_l) = -\infty$$

5. THE SECOND CRITICAL LINE

For each $(\lambda_r, \lambda_l) \in \mathscr{B}$ the invariant probability measures are the convex linear combinations of $\delta_\varnothing$ and $\nu_{\lambda_r, \lambda_l}$.[10] Nevertheless the behavior of the system is not uniform on all of region β as concerns the domains of attraction of the invariant measures. The next two theorems show this fact.

Theorem 7. (*Complete Convergence Theorem*): If $\alpha_1(\lambda_r, \lambda_l) > 0$ and $\alpha_2(\lambda_r, \lambda_l) > 0$, then

$$\forall_\eta \subset \mathbb{Z},\ \xi^\eta_{\lambda_r, \lambda_l}(t) \rightarrow \beta \nu_{\lambda_r},\lambda_l + (1-\beta)\,\delta_\varnothing$$

weakly as $t \rightarrow \infty$, where $\beta = P(\tau^\eta_{\lambda_r, \lambda_l} = \infty)$.

Proof. It is the same as for Theorem 2.28 of Chapter 6 of Ref. 11.

Theorem 8. If $\alpha(\lambda_r, \lambda_l) > 0$ and $\min(\alpha_1(\lambda_r, \lambda_l), \alpha_2(\lambda_r, \lambda_l)) < 0$, then

(a) $\forall \eta \subset \mathbb{Z}$, $|\eta| < \infty$, $\xi^\eta_{\lambda_r, \lambda_l}(t) \rightarrow \delta_\varnothing$, weakly as $t \rightarrow \infty$.
(b) $\exists \eta \subset \mathbb{Z}$ s.t. $\xi^\eta_{\lambda_r, \lambda_l}(t)$ does not converge weakly to any limit as $t \rightarrow \infty$.

Proof.

(a) We must prove that for any finite $A \subset \mathbb{Z}$, $P(\xi^\eta(t) \cap A \neq \varnothing) \to 0$ as $t \to \infty$. We consider the case $\alpha_1(\lambda_r, \lambda_l) > 0$, $\alpha_2(\lambda_r, \lambda_l) < 0$, since the other case is analogous. Define $\zeta = \mathbb{Z} \cap [\min \eta, \infty)$; then

$$P(\xi^\eta(t) \cap A \neq \varnothing) \leqslant P(\xi^\zeta(t) \cap A \neq \varnothing) \leqslant P(\max A \geqslant \min(\xi^\zeta(t)))$$

But by Theorem 3(b), $\min \xi^\zeta(t) \to \infty$ as $t \to \infty$, since $\alpha_2(\lambda_r, \lambda_l) < 0$.

(b) Consider again $\alpha_1(\lambda_r, \lambda_l) > 0$, $\alpha_2(\lambda_r, \lambda_l) < 0$. Since $\alpha(\lambda_r, \lambda_l) > 0$, it follows that $\alpha_1(\lambda_r, \lambda_l) > |\alpha_2(\lambda_r, \lambda_l)|$. Choose $\delta > 0$ such that $\alpha_2(\lambda_r, \lambda_l) + \delta < 0$ and define $a = \alpha_1(\lambda_r, \lambda_l) + \delta$, $b = |\alpha_2(\lambda_r, \lambda_l)| + \delta = |\alpha_2(\lambda_r, \lambda_l) - \delta|$. Observe that $0 < b < a$. Consider now the intervals

$$I_k = \left[-\frac{a^{k+2}}{b^{k+1}}, -\frac{a^{k+1}}{b^k} \right], \qquad k = 0, 1, \ldots$$

and define

$$\eta = \left(\bigcup_{k=1}^{\infty} I_{2k} \right) \cap \mathbb{Z}$$

Consider now the instants $t_k = (a/b)^{k+1}$, $k = 0, 1, \ldots$. We will prove that $P(0 \subset \xi^\eta_{\lambda_r, \lambda_l}(t_{2n+1})) \to 0$ as $n \to \infty$ and $P(0 \subset \xi^\eta_{\lambda_r, \lambda_l}(t_{2n})) \to \rho(\lambda_r, \lambda_l) > 0$ as $n \to \infty$.

By symmetric-duality,

$$P(0 \subset \xi^\eta_{\lambda_r, \lambda_l}(t_k)) = P(\xi^0_{\lambda_r, \lambda_l}(t_k) \cap \eta = \varnothing) \tag{5.1}$$

If $k = 2n + 1$,

$$\begin{aligned}
P(\xi^0_{\lambda_l, \lambda_r}(t_{2n+1}) \cap \eta \neq \varnothing) &\leqslant P([l(\xi^{\mathbb{Z}_+}_{\lambda_l, \lambda_r}(t_{2n+1})), r(\xi^{\mathbb{Z}_-}_{\lambda_l, \lambda_r}(t_{2n+1}))] \not\subset I_{2n+1}) \\
&\leqslant P\left(l_{\lambda_l, \lambda_r}(t_{2n+1}) < -\frac{a^{2n+3}}{b^{2n+2}} \right) + P\left(r_{\lambda_l, \lambda_r}(t_{2n+1}) > -\frac{a^{2n+2}}{b^{2n+1}} \right) \\
&= P\left(\frac{l_{\lambda_l, \lambda_r}(t_{2n+1})}{t_{2n+1}} < -a \right) + P\left(\frac{r_{\lambda_l, \lambda_r}(t_{2n+1})}{t_{2n+1}} > -b \right) \\
&= P\left(\frac{r_{\lambda_r, \lambda_l}(t_{2n+1})}{t_{2n+1}} > \alpha_1(\lambda_r, \lambda_l) + \delta \right) \\
&\quad + P\left(\frac{l_{\lambda_r, \lambda_l}(t_{2n+1})}{t_{2n+1}} < -(\alpha_2(\lambda_r, \lambda_l) + \delta) \right)
\end{aligned}$$

And Theorem 3b implies the two last probabilities above convergence to 0 as $n \to \infty$.

If $k=2n$, write

$$P(\xi^0_{\lambda_l,\lambda_r}(t_{2n})\cap\eta\neq\varnothing)=P(\xi^0_{\lambda_l,\lambda_r}(t_{2n})\neq\varnothing)$$
$$-P(\xi^0_{\lambda_l,\lambda_r}(t_{2n})\neq 0,\ \xi^0_{\lambda_l,\lambda_r}(t_{2n})\cap\eta=\varnothing) \qquad (5.2)$$

But

$$P(\xi^0_{\lambda_l,\lambda_r}(t_{2n})\neq\varnothing,\ \xi^0_{\lambda_l,\lambda_r}(t_{2n})\cap\eta=\varnothing)$$
$$\leqslant P([l(\xi^{\mathbb{Z}_+}_{\lambda_l,\lambda_r}(t_{2n})),\ r(\xi^{\mathbb{Z}_-}_{\lambda_l,\lambda_r}(t_{2n}))]\not\subset I_{2n})$$

And we can prove as before that this probability converges to 0 as $n\to\infty$. Using (5.1) and (5.2), the proof is complete, since $P(\xi^0_{\lambda_l,\lambda_r}(t_{2n})\neq\varnothing)\to\rho(\lambda_l,\lambda_r)=\rho(\lambda_r,\lambda_l)$ as $n\to\infty$. ▮

The conditions of Theorem 7 are satisfied, for instance, by the supercritical BCP; and the conditions of Theorem 8, by the supercritical OSCP.

Theorems 7 and 8 motivate us to define

$$\lambda_{c2}(\theta)=\sup\{\lambda\geqslant 0\colon \min(\alpha_1(\lambda c(\theta),\lambda s(\theta)),\alpha_2(\lambda c(\theta),\lambda s(\theta)))<0\}$$

Proposition 1. (a) If $\theta\in(0,\pi/2)$, then $\lambda_{c1}(\theta)\leqslant\lambda_{c2}(\theta)<\infty$.

(b) If $\theta\in\{0,\pi/2\}$, then $\lambda_{c2}=\infty$.

Proof. (a) The first inequality is trivial. Theorems 6f and 4 imply that, if $\lambda\geqslant\lambda_{c1}(\theta)$, then

$$\bar\alpha_1(\lambda,\theta)=\alpha_1(\lambda c(\theta),\lambda s(\theta))\geqslant(\lambda-\lambda_{c1}(\theta))\,c(\theta)$$

$$\bar\alpha_2(\lambda,\theta)=\alpha_2(\lambda c(\theta),\lambda s(\theta))\geqslant(\lambda-\lambda_{c1}(\theta))\,s(\theta)$$

Therefore, if $\theta\in(0,\pi/2)$, the functions $\lambda\to\bar\alpha_1(\lambda,\theta)$ and $\lambda\to\bar\alpha_2(\lambda,\theta)$ are strictly increasing on $[\lambda_c(\theta),\infty)$ and diverges as $\lambda\to\infty$. So $\lambda_{c2}(\theta)<\infty$.

(b) If $\theta=0$, then $\lambda_r=2\lambda$, $\lambda_l=0$. The construction with the Poisson point process in Sec. 2 yields $l_t\geqslant N_t$, where (N_t) is a Poisson process with rate 1. Then $\alpha^2_t(\lambda_r,\lambda_l)=-El_t\leqslant -t$ and $\alpha_2(\lambda_r,\lambda_l)\leqslant -1$ for any $\lambda\geqslant 0$.

The case $\theta=\pi/2$ is analogous. ▮

Theorem 9. $\theta\to\lambda_{c2}(\theta)$ is a continuous function on $(0,\pi/2)$.

Proof. It is analogous to Theorem 1.

Then $\theta\to\sigma_{c2}(\theta)=2(\sin\theta+\cos\theta)^{-1}\lambda_{c2}(\theta)$ is also continuous on $(0,\pi/2)$ and defines a curve on the phase diagram (λ_r,λ_l).

We define

$$\mathscr{B}_1 = \{(\lambda_r, \lambda_l) \in \mathscr{B}: \lambda < \lambda_{c2}(\theta)\}$$
$$\mathscr{B}_2 = \{(\lambda_r, \lambda_l) \in \mathscr{B}: \lambda > \lambda_{c2}(\theta)\}$$
$$l_2 = \{(\lambda_r, \lambda_l) \in \mathbb{R}^2_+ : \lambda = \lambda_{c2}(\theta)\}$$

Then $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2 \cup l_2 \backslash l_1$. l_2 will be called the second critical line.

At the moment it is clear that on $\mathscr{B}_1$ the conditions of Theorem 8 hold and on $\mathscr{B}_2$ the conditions of Theorem 7 hold (remember that $\lambda \to \bar{\alpha}_i(\lambda, \theta)$ $(i = 1, 2)$ are strictly increasing functions). But we do not know what happens on l_2, nor even if min $(\bar{\alpha}_1(\lambda_{c2}(\theta), \theta)$. $\bar{\alpha}_2(\lambda_{c2}(\theta), \theta)$ is negative, positive, or null. The behavior on l_2 and some of its geometric properties are the main subject of the next sections.

It is easy to identify by elementary methods some subsets of $\mathbb{R}^2_+$ which are contained in $\mathscr{A}$, $\mathscr{B}_1$ or $\mathscr{B}_2$.

We saw already in Sec. 3 that

$$\{(\lambda_r, \lambda_l) \in \mathbb{R}^2_+ : \lambda_r + \lambda_l < 2\} \subset \mathscr{A} \tag{5.3}$$

We can also prove easily that

$$\{(\lambda_r, \lambda_l) \in \mathbb{R}^2_+ : \max(\lambda_r, \lambda_l) < \lambda_{c1}(\pi/4)\} \subset \mathscr{A} \tag{5.4}$$

To do it define $\bar{\lambda} = \max(\lambda_r, \lambda_l)$; then $\rho(\lambda_r, \lambda_l) \leqslant \rho(\bar{\lambda}, \bar{\lambda}) = 0$. Since it is known that $\lambda_c(\pi/4) \geqslant 1.5$ (see Ref. 11, pages 288 and 289), this improves the previous result.

Since r_t increases one unit with rate λ_r and decreases at least one unit with rate 1, $\lambda_r < 1 \Rightarrow r_t \to -\infty$ and $\alpha_1(\lambda_r, \lambda_l) < 0$. But if $\lambda_l > \lambda_c^+ = 2\lambda_{c1}(0)$, then $\alpha(\lambda_r, \lambda_l) > 0$. Therefore

$$\{(\lambda_r, \lambda_l) \in \mathbb{R}^2_+ : \lambda_r < 1 \text{ and } \lambda_l > \lambda_c^+\} \subset \mathscr{B}_1 \tag{5.5}$$

And, of course, we can interchange λ_r and λ_l above.

Finally, it is immediate that

$$\{(\lambda_r, \lambda_l) \in \mathbb{R}^2_+ : \min(\lambda_r, \lambda_l) > \lambda_{c1}(\pi/4)\} \subset \mathscr{B}_2 \tag{5.6}$$

Now it is easy to see that if $\theta < \arctan(1/\lambda_c^+)$, then $\lambda_{c1}(\theta) < \lambda_{c2}(\theta)$. On the other hand, $\lambda_{c1}(\pi/4) = \lambda_{c2}(\pi/4)$. This motivates the definition

$$\theta_c = \inf\{\theta \in [0, \pi/2]: \lambda_{c1}(\theta) = \lambda_{c2}(\theta)\}$$

Then $\arctan(1/\lambda_c^+) \leqslant \theta_c \leqslant \pi/4$. In Sec. 6 we will show that the first inequality is strict. We conjecture, but were not able to prove, that $\theta_c = \pi/4$, i.e., that for any asymmetry ($\theta \neq \pi/4$) there are values of λ such that α_1 and α_2 have opposite signs. Observe that if $\theta_c < \pi/4$, there may in principle be values of θ between θ_c and $\pi/4$ such that $\lambda_{c1}(\theta) < \lambda_{c2}(\theta)$.

6. RELATION WITH ONE-DEPENDENT ORIENTED PERCOLATION

Many important results were obtained for the BCP by Durrett and Griffeath,[3] using a relation between this process and a one-dependent oriented percolation process. Simplified versions of this construction appear in Ref. 11, Chapter 6 and in Ref. 2. In fact this construction can be extended to the asymmetric case easily. Besides consequences which are analogous to the similar statements for the BCP, we obtain results about the geometry of l_1 and l_2 and about the behavior of the process on l_2.

We follow closely the development in Ref. 11. Fix $0 < \beta < \alpha/2$ and $M > 0$ so that $M\beta/2$ and $M\alpha$ are integers. For $(j, k) \in I = \{(j, k) \in \mathbb{Z}^2 : k \geqslant 0$ and $j + k$ is even$\}$, define paralellograms in $\mathbb{Z} \times [0, \infty)$ by

$$L_{jk} = \left\{(x, t) \in \mathbb{Z} \times \left[Mk, M\left(k + 1 + \frac{\beta}{\alpha}\right)\right]: \right.$$
$$\left. 0 \leqslant x + \alpha_2 t - M\left(j\alpha + k\alpha - j\beta + \frac{\beta}{2}\right) \leqslant \beta M\right]$$

$$R_{jk} = \left\{(x, t) \in \mathbb{Z} \times \left[Mk, M\left(k + 1 + \frac{\beta}{\alpha}\right)\right]: \right.$$
$$\left. -\beta M \leqslant x - \alpha_1 t - M\left(j\alpha - k\alpha - j\beta - \frac{\beta}{2}\right) \leqslant 0\right\}$$

See Fig. 2 and compare it with Fig. 2 in Chapter 6 of Ref. 11 (page 296).

The following theorems are then analogous to the corresponding ones in the symmetric case.

Theorem 10. If $(\lambda_r, \lambda_l) \in l_1$, then $\alpha(\lambda_r, \lambda_l) = 0$.

Proof. It is analogous to Corollary 3.20, Chapter 6 of Ref. 11.

Theorem 11. For $(\lambda_r, \lambda_l) \in \mathscr{B}$ and $a < \alpha_1(\lambda_r, \lambda_l)$, $b < \alpha_2(\lambda_r, \lambda_l)$, $\lim_{t \to \infty} (1/t) \log P(r_t < at)$ and $\lim_{t \to \infty} (1/t)\ P(l_t > -bt)$ exist and are strictly negative.

Proof. Analogous to Corollary 3.22, Chapter 6 of Ref. 11.

Theorem 12. If $(\lambda_r, \lambda_l) \in \mathscr{B}$, then there are positive constants C

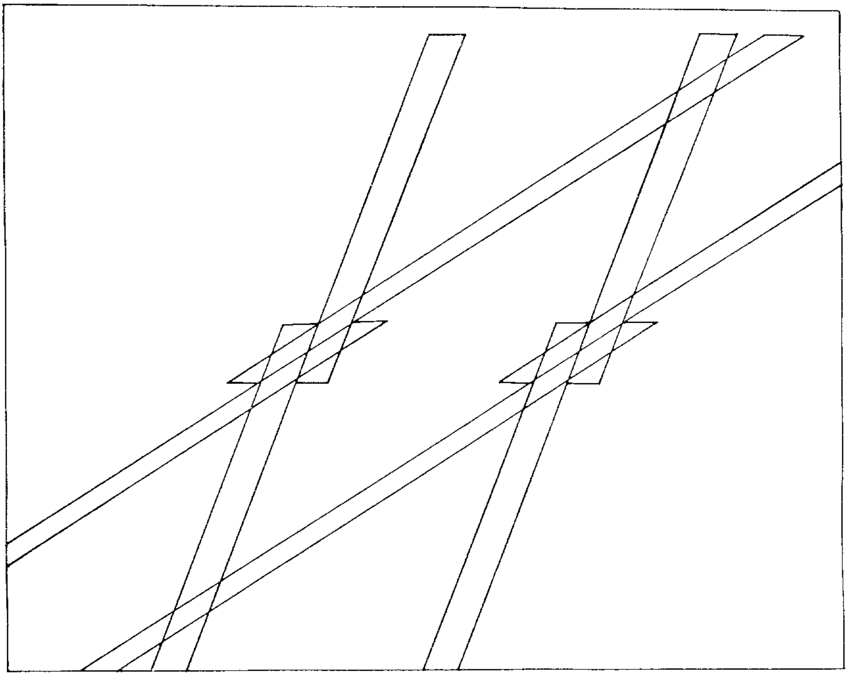

Figure 2

and γ depending only on (λ_r, λ_l) such that for all $t \geqslant 0$ $P(t < \tau^A < \infty) \leqslant C|A|\, e^{-\gamma t}$.

Proof. It is analogous to Theorem 3.23, Chapter 6 of Ref.11.

Theorem 13. If $(\lambda_r, \lambda_l) \in \mathscr{B}$, then there are positive constants C and γ depending only on (λ_r, λ_l) such that for all $t \geqslant 0$ $P(\tau^A < \infty) \leqslant Ce^{-\gamma|A|}$.

Proof. It is analogous to Theorem 3.29, Chapter 6 of Ref. 11.

Theorem 14. If $(\lambda_r, \lambda_l) \in \mathscr{B}$, then $\lim_{t \to \infty} t^{-1}|\xi^0(t)| = 2\alpha(\lambda_r, \lambda_l) \cdot \rho(\lambda_r, \lambda_l)$ a.s. on $[\tau^0 = \infty]$.

Proof. It is analogous to Theorem 3.33, Chapter 6 of Ref. 11.

Theorem 15. $\alpha_1(\lambda_r, \lambda_l)$ and $\alpha_2(\lambda_r, \lambda_l)$ are continuous on $\mathscr{B} \cup l_1 = \mathscr{A}^c$.

Proof. As in the proof of Theorem 1, it is enough to prove the continuity of $x \to \alpha_i(\lambda_r + x,\ \lambda_l + x)$, $x \to \alpha_i(\lambda_r + x, 0)$ and $x \to \alpha_i(0, \lambda_l + x)$ at

$x=0$ for $i=1,2$. These results follow in an analogous way to Theorem 3.36, Chapter 6 of Ref. 11.

We explore now some consequences of these theorems.

Corollary 1. If $(\lambda_r, \lambda_l) \in l_1$, then

(i) $\lambda_r' > \lambda_r \Rightarrow (\lambda_r', \lambda_l) \in \mathscr{B}$

(ii) $\lambda_l' > \lambda_l \Rightarrow (\lambda_r, \lambda_l') \in \mathscr{B}$

Proof. We prove (i). Theorems 4 and 5 imply $\alpha(\lambda_r', \lambda_l) > \alpha(\lambda_r, \lambda_l) = 0$, where the last equality is due to to Theorem 10. Now Theorems 10 and 6c imply $(\lambda_r, \lambda_l') \in \mathscr{B}$. ∎

Therefore no straight line of the form $\lambda_r = \text{const}$ or $\lambda_l = \text{const}$ intercepts l_1 in more than one point.

Corollary 2. $\lambda_c^+ > \lambda_c$ (remember that $\lambda_c = \lambda_{c1}(\pi/4)$ and $\lambda_c^+ = 2\lambda_{c1}(0) = 2\lambda_{c1}(\pi/2)$).

Proof. $(\lambda_c^+, 0) \in l_1$. Then, by Corollary 1, $(\lambda_c^+, \lambda_c^+) \in \mathscr{B}$. Consider the radial family with $\theta = \pi/4$ to finish the proof. ∎

This corollary follows in fact also from Theorem 2 of Ref. 3.

Corollary 3. $\theta_c > \arctan(1/\lambda_c^+)$.

Proof. We know already that $\theta_c \geqslant \arctan(1/\lambda_c^+)$. We have seen also that $\{(\lambda_r, \lambda_l) \in \mathbb{R}_+^2 : \lambda_r > \lambda_c^+,\ \lambda_l < 1\} \subset \mathscr{B}_1$; then

(i) $\theta < \arctan(1/\lambda_c^+) \Rightarrow c(\theta)\,\lambda_{c1}(\theta) \leqslant \lambda_c^+$. Since $\lambda_{c1}(\cdot)$ and $(\cdot)$ are continuous functions,

$$\lambda_{c1}(\arctan(1/\lambda_c^+)) \leqslant \lambda_c^+/c(\arctan(1/\lambda_c^+)) \tag{6.1}$$

(ii) $\theta < \arctan(1/\lambda_c^+) \Rightarrow c(\theta)\,\lambda_{c2}(\theta) \geqslant \lambda_2^+$. Since $\lambda_{c2}(\cdot)$ is also a continuous function,

$$\lambda_{c2}(\arctan(1/\lambda_c^+)) \geqslant \lambda_c^+/c(\arctan(1/\lambda_c^+)) \tag{6.2}$$

If $\theta_c = \arctan(1/\lambda_c^+)$, then it would be necessary for equality to hold in (6.1) and (6.2), and then $(\lambda_c^+, 1) \in l_1 \cap l_2$. Corollary 1 shows that this is in contradiction with the fact that $(\lambda_c^+, 0) \in l_1$.

Corollary 4. $(\lambda_r, \lambda_l) \in l_2 \Leftrightarrow \min(\alpha_1(\lambda_r, \lambda_l), \alpha_2(\lambda_r, \lambda_l)) = 0$.

Proof. $\Rightarrow$) By Theorem 15, for each $\theta \in [0, \pi/2]$, $\lambda \to \bar{\alpha}_1(\lambda, \theta)$ and $\lambda \to \bar{\alpha}_2(\lambda, \theta)$ are continuous functions on $[\lambda_{c1}(\theta), \infty)$. Therefore $\lambda \to \min(\bar{\alpha}_1(\lambda, \theta), \bar{\alpha}_2(\lambda, \theta))$ is also continuous on $[\lambda_{c1}(\theta), \infty)$. On the other hand, by Theorem 10, $\min(\bar{\alpha}_1(\lambda_{c1}(\theta), \theta), \bar{\alpha}_2(\lambda_{c1}(\theta), \theta)) \leqslant 0$, completing the proof.

$\Leftarrow$) Take $\lambda = (\lambda_r + \lambda_l)/2$ and $\theta = \arctan(\lambda_l/\lambda_r)$. Then $\lambda' > \lambda \Rightarrow \min(\bar{\alpha}_1(\lambda', \theta), \bar{\alpha}_2(\lambda', \theta)) > 0$ and $\lambda > \lambda' \Rightarrow \min(\bar{\alpha}_1(\lambda', \theta), \bar{\alpha}_2(\lambda', \theta) < 0$. ∎

Corollary 5. If $(\lambda_r, \lambda_l) \in l_2$, then

(i) $\lambda_r' > \lambda_r \Rightarrow (\lambda_r', \lambda_l) \in \mathscr{B}_2$

(i) $\lambda_l' > \lambda_l \Rightarrow (\lambda_r', \lambda_l) \in \mathscr{B}_2$.

Proof. We prove (i). By Theorem 4 $\alpha_1(\lambda_r', \lambda_l) > \alpha_1(\lambda_r, \lambda_l)$ and by Theorem 5, $\alpha_2(\lambda_r', \lambda_l) > \alpha_2(\lambda_r, \lambda_l)$. Then

$$\min(\alpha_1(\lambda_r', \lambda_l), \alpha_2(\lambda_r', \lambda_l)) > \min(\alpha_1(\lambda_r, \lambda_l), \alpha_2(\lambda_r, \lambda_l)) = 0 \tag{6.3}$$

where the last equality follows from Corollary 4 above. Now (6.3) implies by the definition of $\mathscr{B}_2$ and l_2 that $(\lambda_r', \lambda_l) \in \mathscr{B}_2 \cup l_2$, and Corollary 4 implies that $(\lambda_r', \lambda_l) \notin l_2$. ∎

Therefore no straight line of the form $\lambda_r = \text{const}$ or $\lambda_l = \text{const}$ intercepts l_2 in more than one point.

7. STATIONARY DISTRIBUTION FOR THE EDGE PROCESS AND SOME OF ITS APPLICATIONS

Notation:

$E = \{\eta \subset \mathbb{Z}: r(\eta) < \infty, |\eta| = \infty\}$.
$\tilde{E} = \{\eta \in E: r(\eta) = 0\}$.
Given $x \in \mathbb{Z}$, $\eta + x = \{y \in \mathbb{Z}: y - x \in \eta\}$.
$S: E \to \tilde{E}$ defined by $S(\eta) = \eta - r(\eta)$.

Given a measure μ with support on E,

$$r^{\mu}_{\lambda_r, \lambda_l}(t) = r(\xi^{\mu}_{\lambda_r, \lambda_l}(t)).$$

When no confusion is possible, we will write r^{μ}_t instead of $r^{\mu}_{\lambda_r, \lambda_l}(t)$.

Durrett[2] proved in the symmetric case, for $\lambda \geqslant \lambda_c$, the existence of a measure μ concentrated on $\tilde{E}$ such that $(S\xi^{\mu}_{\lambda_r, \lambda_l}(t))$ is a stationary process (in fact he stated the theorem there for a discrete time analogue of the BCP, but the proof for the BCP is essentially the same). Galves and Presutti[4] improved this result, showing in particular that for any initial configuration η, $S\xi^{\eta}_{\lambda_r, \lambda_l}(t) \to \mu$ weakly as $t \to \infty$, for $\lambda > \lambda_c$.

The techniques in Refs. 2 and 4 work also in the asymmetric case, giving:

Theorem 16. Suppose $(\lambda_r, \lambda_l) \in \mathscr{B} \cup l_1$. Then there exists a probability measure $\mu = \mu_{\lambda_r, \lambda_l}$ with support on $\tilde{E}$ such that

(a) $(S\xi^{\mu}_{\lambda_r,\lambda_l}(t), t \geqslant 0)$ is a stationary process.

(b) $(r(\xi^{\mu}_{\lambda_r,\lambda_l}(t)), t \geqslant 0)$ is a process with stationary increments.

(c) $E(r(\xi^{\mu}_{\lambda_r,\lambda_l}(1))) = \alpha_1(\lambda_r, \lambda_l)$.

(d) If $(\lambda_r, \lambda_l) \in \mathscr{B}$, then $\forall \eta \in \tilde{E}$, $S\xi^{\mu}_{\lambda_r,\lambda_l}(t) \to \mu$ weakly as $t \to \infty$.

Remark. (d) implies that on $\mathscr{B}$, μ is the only measure with property (a). We do not know if this is true on l_1.

In the remainder of this section we apply these facts to prove that as $\theta \to 0$, the line l_2 is asymptotic to the straight line $\lambda_l = 1$. Some of the intermediate results are of interest for their own right. For this purpose it is better to use the cartesian coordinates (λ_r, λ_l) on the phase diagram instead of(θ, λ).

Definitions.

$$\bar{\lambda}_r(\lambda_l) = \inf\{\lambda_r \in \mathbb{R}_+ : (\lambda_r, \lambda_l) \in \mathscr{B}_2\}$$

$$\phi = \inf\{\lambda_l > 0 : \bar{\lambda}_r(\lambda_l) < \infty\}$$

We know already from (5.5) that $\phi \geqslant 1$.

Proposition 2. If $\bar{\lambda}_r(\lambda_l) < \infty$, then $(\bar{\lambda}_r(\lambda_l), \lambda_l) \in l_2$. If $\bar{\lambda}_r(\lambda_l) = \infty$, then there is no $\lambda_r \in \mathbb{R}_+$ such that $(\lambda_r, \lambda_l) \in l_2$.

Proof. In a fashion analogous to Corollary 4, one can prove that

$$\min(\alpha_1(\lambda_r, \lambda_l), \alpha_2(\lambda_r, \lambda_l)) = 0 \Leftrightarrow \lambda_r = \bar{\lambda}_r(\lambda_l)$$

Using Corollary 4 we complete the proof. ∎

Proposition 3. (a) $\lambda_l \to \bar{\lambda}_r(\lambda_l)$ is a strictly decreasing function on (ϕ, ∞).

(b) $\lim_{\lambda_l \to \phi^+} \bar{\lambda}_r(\lambda_l) = \infty$.

Proof.

(a) Take $\lambda_l' > \lambda_l > \phi$. Then $\bar{\lambda}_r(\lambda_l) < \infty$. Thus $(\bar{\lambda}_r(\lambda_l), \lambda_l) \in l_2$ and Corollary 5 implies that $(\bar{\lambda}_r(\lambda_l), \lambda_l') \in \mathscr{B}_2$. Since $(\bar{\lambda}_r(\lambda_l'), \lambda_l') \in l_2$, $\bar{\lambda}_r(\lambda_l') < \lambda_r(\lambda_l)$.

(b) Part (a) implies that $\bar{\lambda}_r(\lambda_l) \nearrow a \in (-\infty, \infty]$ as $\lambda_l \searrow \phi$. Suppose $a < \infty$. Then for $b > a$ and $\theta = \arctan(\phi/b)$, we would have

(i) if $\lambda < (\phi + b)/2$, then $(\lambda c(\theta), \lambda s(\theta)) \notin l_2$ since $\bar{\lambda}_r(\lambda_l) = \infty$ for $\lambda_l < \phi$.

(ii) if $\lambda \geqslant (\phi + b)/2$, then $(\lambda c(\theta), \lambda s(\theta)) \notin l_2$ since for $\lambda_l \geqslant \phi$, $\bar{\lambda}_r(\lambda_l) \leqslant a < b$.

Therefore $\lambda_{c2}(\theta) = \infty$. Since $\phi \geqslant 1$, this contradicts Proposition 1. ∎

We will use now Theorem 16 in order to prove that $\phi \leqslant 1$.

Definition. For fixed $(\lambda_r, \lambda_l) \in \mathscr{B} \cup l_1$, take a random configuration $\eta \in \tilde{E}$ with distribution $\mu_{\lambda_r, \lambda_l}$. Define now the random variables

$$X_0 = 0$$

$$X_i = -\sup\{x < X_{i-1} : x \in \eta\}$$

We will use also the notation $E_{\lambda_r, \lambda_l}(X_i) = \sum_{x=0}^{\infty} x\mu_{\lambda_r, \lambda_l}(X_i = x)$, or just $E(X_i)$ if no confusion is possible.

In order to prove the next lemma we introduce now a different construction of the processes $(\xi^{\eta}_{\lambda_r, \lambda_l}(t),\ t \geqslant 0)$. Given $u > 0$, k positive integer and $\eta \in E$, we define the random function $t \to \hat{\xi}^{(\eta, ku)}_{\lambda_r, \lambda_l}(t)$ from $[ku, \infty)$ to E in the following way:

(i) $\hat{\xi}^{(\eta, ku)}_{\lambda_r, \lambda_l}(ku) = \eta$

(ii) if $nu \leqslant t \leqslant (n+1)u$, with $n \geqslant k$, then

$$\hat{\xi}^{(\eta, ku)}_{\lambda_r, \lambda_l}(t) = \xi^{(S\zeta, nu)}_{\lambda_r, \lambda_l}(t) + r(\zeta)$$

where $\zeta = \hat{\xi}^{(\eta, ku)}_{\lambda_r, \lambda_l}(t)$.

As before, we will omit λ_r, λ_l if this does not lead to confusion and abbreviate $\hat{\xi}^{(\eta, 0)}(t) = \hat{\xi}^{\eta}(t)$ and $\hat{\xi}^{(\{x\}, ku)}(t) = \hat{\xi}^{(x, ku)}(t)$.

The fact that $(\xi^{\eta}_{\lambda_r, \lambda_l}(t))$ and $(\hat{\xi}^{\eta}_{\lambda_r, \lambda_l}(t))$ hve the same law follows immediately from the properties of the Poisson processes with which they are constructed.

This construction was introduced in Ref. 4 and as we shall see is very suitable for some couplings.

Lemma 1.

(*a*) If $(\lambda_r, \lambda_l) \in \mathscr{B}$, then $\mu_{\lambda_r, \lambda_l}(X_k > kd) \leqslant Ck^{3/2} e^{-\gamma d^{1/2}}$, where C and γ depend only on λ_r and λ_l.

(b) For fixed $(\lambda_r, \lambda_l) \in \mathscr{B}$, $\mu_{\lambda_r, \lambda'_l}(X_k > kd) \leqslant Ck^2 e^{-\gamma d^{1/2}}$ for any $\lambda'_l \geqslant \lambda_l$, where C and γ depend only on λ_r and λ_l.

(c) $E_{\lambda_r, \lambda_l}(X_k) \leqslant Ck^{5/2}$, where C depends only on λ_r and λ_l.

Proof.

(a) Take $t = dk/2(\lambda_r + \lambda_l)$ and u as the integer such that $u^2 \leqslant t < (u+1)^2$. Define the random variable

$$L = \inf\{l: \hat{\xi}^{(0, lu)}_{\lambda_r, \lambda_l}(l+1) \neq \varnothing\}$$

Define also the events

$$E_1(l) = [r(\xi^{(Z_-, lu)}(t)) \in [\alpha_1(t - lu)/2, 2\lambda_r(t - lu)]]$$
$$E_2(l) = [l(\xi^{(Z_+, lu)}(t)) \in [2\lambda_l(t - lu), \alpha_2(t - lu)/2]]$$
$$E(l) = E_1(l) \cap E_2(l)$$
$$F_i(l) = [\xi^{(Z, lu)}(t) \cap A_i(l) \neq \emptyset]$$

where

$$A_i = \left(\frac{\alpha_1 + \alpha_2}{8} \frac{t}{k} i, \frac{\alpha_1 + \alpha_2}{8} \frac{t}{k} (i+1)\right]$$
$$= \left(\frac{(\alpha_1 + \alpha_2) d}{16(\lambda_r + \lambda_l)} i, \frac{(\alpha_1 + \alpha_2) d}{16(\lambda_r + \lambda_l)} (i+1)\right]$$
$$A_i(l) = A_i - \alpha_2(t - lu)/2 = \{x \in \mathbb{R}: x + \alpha_2(t - lu)/2 \in A_i\}$$
$$F(l) = \bigcap_{i=0}^{k-1} F_i(l)$$

We consider the contact process starting from the distribution μ. Then $S\xi^\mu(t)$ has also distribution μ and we will consider η in the definition of X_k as $S\xi^\mu(t)$. Then if v is the largest integer such that $uv < t/2$,

$$P(X_k > kd) \leq \sum_{l=0}^{v} P(L = l, X_k > kd) + P(Lu > t/2) \tag{7.1}$$

But, recalling the notation $\rho = \rho(\lambda_r, \lambda_l) = P(\tau^0 = \infty) > 0$,

$$P(Lu > t/2) \leq (1 - \rho)^{v-1} \leq Ce^{-\gamma t^{1/2}} \leq Ce^{-\gamma d^{1/2}} \tag{7.2}$$

And

$$\begin{aligned} P(L = l, X_k > kd) &= P(L = l, X_k > kd, \tau^{(0, lu)} < \infty) \\ &+ P(L = l, X_k > kd, \tau^{(0, lu)} = \infty) \leq P(u(l+1) < \tau^{(0, lu)} < \infty) \\ &+ P(X_k > kd, \tau^{(0, lu)} = \infty) \end{aligned} \tag{7.3}$$

The first term above can be controlled using Theorem 12

$$P(u(l+1) < \tau^{(0, lu)} < \infty) = P(u < \tau^0 < \infty) \leq Ce^{-\gamma u} \leq Ce^{-\gamma t^{1/2}} \leq Ce^{-\gamma d^{1/2}} \tag{7.4}$$

For the other term in (7.3) we write

$$\begin{aligned} P(X_k > kd, \tau^{(0, lu)} = \infty) &\leq P(E^c(l)) + P(F^c(l)) \\ &+ P(X_k > kd, \tau^{(0, lu)} = \infty, E(l), F(l)) \end{aligned} \tag{7.5}$$

and note that by Theorem 11 and exponential bounds for Poisson processes,

$$P(E^c(l)) \leqslant Ce^{-\gamma(t-lu)} \leqslant Ce^{-\gamma t/2} \leqslant Ce^{-\gamma d} \tag{7.6}$$

And by symmetric-duality and Theorem 13

$$P(F_i^c(l)) = P(\xi^{A_i}_{\lambda_l,\lambda_r}(t-lu) = \varnothing) \leqslant P(\tau^{A_i}_{\lambda_l,\lambda_r} < \infty) \leqslant Ce^{-\gamma|A_i|} = Ce^{-\gamma d}$$

Thus

$$P(F^c(l)) \leqslant Cke^{-\gamma d} \tag{7.7}$$

Finally note that the event $[X_k > kd,\ \tau^{(0,lu)} = \infty,\ E(l),\ F(l)]$ is void. This follows from P3 in Sec. 2 and the way we defined $E(l)$ and $F(l)$. Combining (7.1) to (7.7) yields

$$P(X_k > kd) \leqslant \sum_{l=0}^{v} Cke^{-\gamma d^{1/2}} + Ce^{-\gamma d^{1/2}} \leqslant Ck^{3/2}e^{-\gamma d^{1/2}}$$

(b) Take u, v, t, and L as above. For typographical reasons we write $\mu = \mu_{\lambda_r,\lambda_l}$, $\mu' = \mu_{\lambda_r,\lambda_l'}$. Define on the same probability space, as in the proof of P8 in Sec. 2, $(\xi^{\mu}_{\lambda_r,\lambda_l}(s))$ and $(\xi^{\mu'}_{\lambda_r,\lambda_l'}(s))$. The initial measures μ and μ' can be taken independently. We define now the random configurations $\eta(s)$ and $\eta'(s)$ by

$$\text{if } s \leqslant Lu, \quad \begin{cases} \eta(s) = \xi^{\mu}_{\lambda_r,\lambda_l}(s) \\ \eta'(s) = \xi^{\mu'}_{\lambda_r,\lambda_l'(s)} \end{cases}$$

$$\text{if } s > Lu, \quad \begin{cases} \eta(s) = \xi^{(\zeta,Lu)}_{\lambda_r,\lambda_l}(s) & \text{where} \quad \zeta = \xi^{\mu}_{\lambda_r,\lambda_l}(Lu) \\ \eta'(s) = \xi^{(\zeta',Lu)}_{\lambda_r,\lambda_l}(s) & \text{where} \quad \zeta' = \xi^{\mu'}_{\lambda_r,\lambda_l'}(Lu) \end{cases}$$

Then $S\eta(s) =^D \mu$ and $S\eta'(s) =^D \mu'$ (where $=^D$ means equal in distribution).

The important property of the coupling above is that on the events $[L = l,\ \tau^{(0,lu)} = \infty]$ the following holds for $t > l_u$:

$$\eta(t) \cap R_l \subset \eta'(t) \cap R_l \tag{7.8}$$

where $R_l = [l(\xi^{(Z_+,lu)}_{\lambda_r,\lambda_l}(t)),\ r(\xi^{(Z_-,lu)}_{\lambda_r,\lambda_l}(t))]$.

Define now the events

$$G(l) = [r(\eta'(t)) - l(\xi^{(Z_+,lu)}_{\lambda_r,\lambda_l}(t)) < 2(\lambda_r - \lambda_l)\, t]$$

Below we use the notation $\alpha_i = \alpha_i(\lambda_r, \lambda_l)$, where $i = 1, 2$ or nothing.

$$P(X'_k > kd) \leqslant \sum_{l=0}^{v} P(L = l, X'_k > kd) + P(Lu > t) \tag{7.9}$$

$$\begin{aligned} P(L = l, X'_k > kd) \leqslant & P(u(l+1) < \tau^{(0,lu)} < \infty) \\ & + P(L = l, X'_k > kd, \tau^{(0,lu)} = \infty) \end{aligned} \tag{7.10}$$

$$\begin{aligned} P(L = l, X'_k > kd, \tau^{(0,lu)} = \infty) \leqslant & P(r(\xi^{(Z^-_-,lu)}_{\lambda_r,\lambda_l}(t)) > 2\lambda_r t) \\ & + P(l(\xi^{(Z_+,lu)}_{\lambda_r,\lambda_l}(t)) < 2\lambda_l t) \\ & + P(L = l, X'_k > kd, \tau^{(0,lu)} = \infty, G(l)) \end{aligned} \tag{7.11}$$

From exponential bounds for Poisson processes the two first terms above are smaller than $Ce^{-\gamma(t-lu)} \leqslant Ce^{-\gamma d}$. The last one can be controlled as follows:

$$\begin{aligned} & P(L = l, X'_k > kd, \tau^{(0,lu)} = \infty, G(l)) \\ & \quad \leqslant P(L = l, \tau^{(0,lu)} = \infty, |\eta(t) \cap R_l| < k) \\ & \quad + P(L = l, X'_k > kd, \tau^{(0,lu)} = \infty, G(l), |\eta(t) \cap R_l| \geqslant k) \end{aligned} \tag{7.12}$$

But (7.8) implies that

$$[L = l, \tau^{(0,lu)} = \infty, G(l), |\eta(t) \cap R_l| \geqslant k] \subset [X'_k \leqslant 2(\lambda_r + \lambda_l)\, t = kd]$$

and the last probability in the r.h.s. of (7.12) is zero. For the other, write

$$\begin{aligned} & P(|\eta(t) \cap R_l| < k, L = l, \tau^{(0,lu)} = \infty) \\ & \quad \leqslant P(l(\xi^{(Z_+,lu)}_{\lambda_r,\lambda_l}(t)) \geqslant -\alpha_2 t/4) + P(r(\xi^{(Z_-,lu)}_{\lambda_r,\lambda_l}(t)) \leqslant \alpha_1 t/4) \\ & \quad + P(r(\xi^{(Z_-,lu)}_{\lambda_r,\lambda_l}(t)) - l(\xi^{(Z_+,lu)}_{\lambda_r,\lambda_l}(t)) > \alpha t/2, |\eta(t) \cap R_l| < k, L = l, \tau^{(0,lu)} = \infty) \end{aligned} \tag{7.13}$$

By Theorem 11, the two first terms above are smaller than $Ce^{-\gamma(t-lu)} \leqslant Ce^{-\gamma d}$. And using part (a) of the theorem, the last term is bounded above by

$$P(X_k > \alpha t/2) = P\left(X_k > \frac{\alpha kd}{4(\lambda_r + \lambda_l)}\right) \leqslant Ck^{3/2} e^{-\gamma d^{1/2}} \tag{7.14}$$

The thesis follows now from (7.9) to (7.14). Note that the extra factor $k^{1/2}$ comes from the sum in (7.9).

(c) Using part (a) above,

$$E(X_k) = kE(X_k/k) = k \int_0^\infty P(X_k/k > x)\, dx \leqslant k \int_0^\infty Ck^{3/2} e^{-\gamma x^{1/2}}\, \mathrm{dx} \leqslant Ck^{5/2} \quad \blacksquare$$

Theorem 17. If λ_r is fixed and $\lambda_l \to \infty$, then

(a) $X_1 \to 1$ in probability.

(b) $E_{\lambda_r, \lambda_l}(X_1) \to 1$.

Proof.

(a) For any $t \geqslant 0$, $S\xi^\mu(t) =^D \mu$. Take $t = (\lambda_l)^{-1/2}$; then

$$P(X_1 = 1) P(-1 \in S\xi^\mu(t)) \geqslant P(U_1^{-1} > t, U_1^0 > t, U_1^{(0,1)} > t, U_1^{(0,-1)} < t)$$
$$= e^{-(2+\lambda_r)t}(1 - e^{-\lambda_l t})$$

which goes to 1 as $\lambda_l \to \infty$.

(b) For any λ_r, $(\lambda_r, 4) \in \mathscr{B}$. Given $\varepsilon > 0$, by Lemma 1b, if $\lambda_l > 4$, then $\mu_{\lambda_r, \lambda_l}(X_1 > x) \leqslant Ce^{-\gamma x^{1/2}}$, where C and γ depend on λ_r but not on λ_l. Thus it is possible to choose $D > 0$ such that, for any $\lambda_l \geqslant 4$,

$$\int_D^\infty \mu_{\lambda_r, \lambda_l}(X_1 > x)\, dx \leqslant \frac{\varepsilon}{2}$$

But

$$E(X_1) = \int_0^\infty \mu_{\lambda_r, \lambda_l}(X_1 > x)\, dx \leqslant \mu_{\lambda_r, \lambda_l}(X_1 = 1)$$
$$+ \mu_{\lambda_r, \lambda_l}(X_1 > 1) \cdot D + \int_D^\infty \mu_{\lambda_r, \lambda_l}(X_1 > x)\, dx$$

Using part (a) of the present theorem, if λ_l is large enough, then $\mu_{\lambda_r, \lambda_l}(X_1 > 1) < \varepsilon/2D$. Thus $E(X_1) \leqslant 1 + \varepsilon$. ∎

Definitions. $m(\lambda_r, \lambda_l) = E_{\lambda_r, \lambda_l}(X_1)$. We will write just m if no confusion is possible.

$$r^\mu_{\lambda_r, \lambda_l}(t) = r(\xi^\mu_{\lambda_r, \lambda_l(t)})$$

As before, we will write sometimes just r_t^μ.

The following theorem is intuitively clear (but not so easy to prove): r_t^μ increases one unit at rate λ_r and decreases in mean $m(\lambda_r, \lambda_l)$ at rate 1.

Theorem 18. Consider $(\lambda_r, \lambda_l) \in \mathscr{B}$. Then $\alpha_1(\lambda_r, \lambda_l) = \lambda_r - m(\lambda_r, \lambda_l)$.

Proof. By Theorems 16b and c, $Er_t^\mu = t \cdot \alpha_1$. Therefore

$$\alpha_1 = \lim_{t \to 0} \frac{Er_t^\mu}{t} = \frac{d}{dt} Er_t^\mu \tag{7.15}$$

If $r(\cdot)$ were a cylindrical function, it would be enough to do now a calculation with the generator. Since this is not the case, we will use some comparison processes. Consider the DPS with which $(\xi^{\mu}(t))$ is constructed, and define, for each $\eta \in \tilde{E}$, η_t by:

(i) If $U_1^0 > t$, $U_1^1 > t$ and $U_1^{(0,1)} \leqslant t$, then $\eta_t = 1$

(ii) If $U_1^0 > t$ and $U_1^1 \leqslant t$, then $\eta_t = 0$

(iii) If $U_1^0 > t$ and $U_1^{(0,1)} > t$, then $\eta_t = 0$

(iv) If $U_1^0 \leqslant t$, then $\eta_t = \sup\{x \in \eta \colon U_1^x > t\}$

Then, almost surely, $r(\eta_t) \leqslant r(\xi^{\eta}(t)) = r_t^{\eta}$. The distribution of $r(\eta_t)$ can be easily obtained:

$$P(r(\eta_t) = 1) = (e^{-t})^2(1 - e^{-\lambda_r t})$$

$$P(r(\eta_t) = 0) = (e^{-t})(1 - e^{-t}) + e^{-t}e^{-\lambda_r t}$$

$$P(r(\eta_t) = -X_k(\eta)) = (1 - e^{-t})^k e^{-t}, \qquad k = 1, 2, \ldots$$

where $X_0(\eta) = 0$, $X_k(\eta) = -\sup\{x \in \eta \colon x \leqslant X_{k-1}(\eta)\}$, $k = 1, 2, \ldots$. Then

$$E(r_t^{\eta}) \geqslant E(\eta_t) = (1 - e^{-\lambda_r t})\, e^{-2t} - \sum_{n=1}^{\infty} (1 - e^{-t})^n e^{-t} X_n(\eta)$$

Using monotone convergence

$$E(r_t^{\mu}) \geqslant \int E(\eta_t)\, d\mu(\eta) = (1 - e^{-\lambda_r t})\, e^{-2t} - \sum_{n=1}^{\infty} (1 - e^{-t})^n e^{-t} \int X_n(\eta)\, d\mu(\eta) \tag{7.16}$$

Using Lemma 1c,

$$\int X_n(\eta)\, d\mu(\eta) = E_{\lambda_r, \lambda_l}(X_n) \leqslant Cn^{5/2}$$

Thus

$$\sum_{n=2}^{\infty} (1 - e^{-t})^n e^{-t} \int X_n(\eta)\, d\mu(\eta) \to 0$$

as $t \to 0$, and by (7.16)

$$\liminf_{t \to 0} \frac{E(r_t^{\mu})}{t} \geqslant \lambda_r - m(\lambda_r, \lambda_l) \tag{7.17}$$

Now define for, each $\zeta \in \tilde{E}$, ζ_t by

(i) If $U_1^{(0,1)} \leqslant t$, then $\zeta_t = \inf\{n \geqslant 0: U_1^{(n,n+1)} > t\}$

(ii) If $U_1^{(0,1)} > t$ and $U_1^0 > t$, then $\zeta_t = 0$

(iii) If $U_1^{(0,1)} > t$, $U_1^0 \leqslant t$, and $U_1^{(X_1(\xi), X_1(\xi)+1)} > t$, then $\zeta_t = -X_1(\zeta)$

(iv) Otherwise $\zeta_t = 0$

Then, almost surely, $r(\zeta_t) \geqslant r(\xi^\zeta(t)) = r_t^\zeta$. Therefore

$$E(r_t^\zeta) \leqslant E(r(\zeta_t)) = -e^{-2\lambda_r t}(1-e^{-t})\, X_1(\zeta) + \sum_{n=1}^{\infty} (1-e^{-\lambda_r t})^n\, e^{-\lambda_r t}\, n$$

And as before we can conclude that

$$\limsup_{t \to 0} \frac{E(r_t^\mu)}{t} \leqslant \lambda_r - m(\lambda_r, \lambda_l) \tag{7.18}$$

The theorem follows from (7.15), (7.17), and (7.18). ∎

The following theorem shows that l_2 is asymptotic to the straight lines $\lambda_l = 1$ and $\lambda_r = 1$.

Theorem 19. $\phi = 1$.

Proof. Using Theorem 18, we obtain

$$\alpha_1(\lambda_r, \lambda_l) = \alpha_1(\lambda_l, \lambda_r) = \lambda_l - m(\lambda_l, \lambda_r)$$

By Theorem 17, if $\lambda_l > 1$ is fixed and $\lambda_r \to \infty$, then

$$\alpha_2(\lambda_r, \lambda_l) \to \lambda_l - 1 > 0$$

If $\lambda_r > 4$, then by Theorem 4,

$$\alpha_1(\lambda_r, \lambda_l) \geqslant \alpha_1(4, \lambda_l) + (\lambda_r - 4)$$

Since $(4, \lambda_l) \in \mathscr{B}$, $\alpha_1(4, \lambda_l) > -\infty$ and

$$\alpha_1(\lambda_r, \lambda_l) \to \infty$$

as $\lambda_r \to \infty$. Therefore, if $\lambda_l > 1$, $\bar{\lambda}_r(\lambda_l) < \infty$ and $\phi \leqslant 1$. Since we know already that $\phi \geqslant 1$, the proof is complete. ∎

8. BEHAVIOR ON $l_2 \backslash l_1$

Remember that

$$l_2 \backslash l_1 = \{(\lambda_r, \lambda_l) \in \mathbb{R}_+^2 : \max(\alpha_1, \alpha_2) > 0,\ \min(\alpha_1, \alpha_2) = 0\}$$

On these points Theorems 7 and 8 do not apply. In their place we have

Theorem 20. If $\alpha_1(\lambda_r, \lambda_l) > 0$ (resp., $\alpha_2(\lambda_r, \lambda_l) > 0$) and $\alpha_2(\lambda_r, \lambda_l) = 0$ (resp., $\alpha_1(\lambda_r, \lambda_l) = 0$), then

(a) for $|\eta| < \infty$, $\xi^\eta(t) \to (1/2)\, P(\tau^\eta = \infty)\, \nu + (1 - (1/2)\, P(\tau^\eta = \infty))\, \delta_\varnothing$, as $t \to \infty$.

(b) for $l(\eta) > -\infty$ (resp., $r(\eta) < \infty$) and $|\eta| = \infty$, $\xi^\eta(t) \to (1/2)\, \nu + (1/2)\, \delta_\varnothing$, as $t \to \infty$.

(c) there are $\eta \subset \mathbb{Z}$ such that $\xi^\eta(t)$ does not converge as $t \to \infty$.

The proof of this theorem is based on the following theorem, proved in Ref. 4 for the symmetric case and easily extended to the asymmetric one.

Theorem 21. If $(\lambda_r, \lambda_l) \in \mathscr{B}$, then for any η such that $|\eta| = \infty$, $r(\eta) < \infty$ and any ζ such that $|\zeta| = \infty$, $l(\zeta) < \infty$ the processes $(\varepsilon(r^\eta_{t/\varepsilon^2} - \alpha_1 \varepsilon^{-2} t),\ t \geqslant 0)$ and $(\varepsilon(l^\zeta_{t/\varepsilon^2} + \alpha_2 \varepsilon^{-2} t),\ t \geqslant 0)$ converge in law when $\varepsilon \searrow 0$ to Brownian motions which have strictly positive diffusion coefficients, respectively σ_r^2 and σ_l^2 independent of η and ζ.

From Theorem 21 it is clear why Theorem 20 should hold: one edge spreads out while the other fluctuates randomly past finite sites; hence the 1/2.

Proof of Theorem 20.

(a) We must prove that for any $A \subset \mathbb{Z}$ such that $|A| < \infty$,

$$P(\xi^\eta(t) \cap A \neq \varnothing) \to (1/2)\, P(\tau^\eta = \infty)\, \nu(\zeta: \zeta \cap A \neq \varnothing) \quad \text{as} \quad t \to \infty$$

Notation: $a = l(\eta) - 2\lambda_l t^{1/3}$, $b = r(\eta) + 2\lambda_r t^{1/3}$, $c = l(A) - 2\lambda_r t^{1/3}$, $d = r(A) + 2\lambda_l t^{1/3}$. For each $x \in \mathbb{Z}$, $\zeta_x^+ = [x, \infty) \cap \mathbb{Z}$ and $\zeta_x^- = (-\infty, x] \cap \mathbb{Z}$.

Consider the events

$$\begin{aligned}
E &= [\xi^\eta(t) \cap A \neq \varnothing] \\
F_1 &= [\tau^\eta > t^{1/3}] \\
F_2 &= [\xi^{(\mathbb{Z}, t - t^{1/3})}(t) \cap A \neq \varnothing] \\
G_1 &= [\xi^\eta(t^{1/3}) \subset [a, b]] \\
G_2 &= [\xi^{(\zeta_c^- \cup \zeta_d^+, t - t^{1/3})}(t) \cap A = \varnothing] \\
H &= [l(\xi^{(\zeta_a^+, t^{1/3})}(t - t^{1/3})) > d] \\
I &= [l(\xi^{(\zeta_b^+, t^{1/3})}(t - t^{1/3}) < c] \\
J &= [r(\xi^{(\zeta_a^-, t^{1/3})}(t - t^{1/3}) > d] \\
K_1 &= [\tau^\eta > t] \\
K_2 &= [\xi^{\mathbb{Z}}(t) \cap A \neq \varnothing]
\end{aligned}$$

Then

$$P(E) = P(EF_1F_2) \leqslant P(G_1^c) + P(G_2^c) + P(F_1F_2H^c) + P(EG_1G_2H)$$

But $[EG_1G_2H] = \varnothing$ and by the properties of Poisson Processes, $P(G_1^c) \to 0$ and $P(G_2^c) \to 0$ as $t \to \infty$. Also $P(F_1F_2H^c) = P(F_1) \cdot P(F_2) \cdot P(H)$, and as $t \to \infty$,

$$P(F_1) \to P(\zeta^\eta = \infty)$$
$$P(F_2) \to \nu(\zeta : \zeta \cap A \neq \varnothing)$$
$$P(H) \to 1/2$$

where the last limit follows from Theorem 21. Thus

$$\limsup_{t \to \infty} P(E) \leqslant (1/2)\, P(\zeta^\eta = \infty)\, \nu(\zeta : \zeta \cap A \neq \varnothing)$$

On the other hand,

$$P(E) \geqslant P(EG_1G_2IJK_1K_2) = P(G_1G_2IJK_1K_2) - P(E^cG_1G_2IJK_1K_2) \qquad (8.1)$$

But from the nearest-neighbor nature of the interaction, it follows that $[E^cG_1G_2IJK_1K_2] = \varnothing$. For the other term, we write

$$P(G_1G_2IJK_1K_2) = P(F_1F_2I) - P(F_1F_2I(G_1G_2JK_1K_2)^c) \qquad (8.2)$$

and

$$P(F_1F_2I) = P(F_1)\, P(F_2)\, P(I) \qquad (8.3)$$

$$P(F_1F_2I(G_1G_2JK_1K_2)^c) \leqslant P(F_1K_1^c) + P(F_2K_2^c) + P(G_1^c) + PG_2^c) + P(J^c) \qquad (8.4)$$

Theorem 11 implies that $P(J^c) \to 0$, as $t \to \infty$.
Theorem 12 implies that $P(F_iK_i^c) \to 0$ as $t \to \infty$ $(i = 1, 2)$.
Theorem 21 implies tat $P(I) \to 1/2$ as $t \to \infty$.
Then (8.1) to (8.4) imply that

$$\liminf_{t \to \infty} P(E) \geqslant (1/2)\, P(\tau^\eta = \infty)\, \nu(\zeta : \zeta \cap A \neq \varnothing)$$

(b) It is analogous to (a).

(c) Choose $a > \alpha_1$ and define the t_i, $i = 0, 1, 2,\ldots$ by

$$t_0 = 1, \qquad t_{i+1} = (at_i)^3$$

Define the intervals $I_k = [-at_k, -t_k^{1/3}]$. And consider the configuration

$$\eta = \left(\bigcup_{k=1}^{\infty} I_{2k} \right) \cap \mathbb{Z}$$

Then

$$P(0 \subset \xi^n(t_{2n})) \to \rho(\lambda_r, \lambda_l) \qquad \text{as} \quad n \to \infty \tag{8.5}$$

and

$$P(0 \subset \xi^n(t_{2n+1})) \to (1/2)\, \rho(\lambda_r, \lambda_l) \qquad \text{as} \quad n \to \infty \tag{8.6}$$

Since $\rho(\lambda_r, \lambda_l) > 0$ on $l_2 \backslash l_1$, this is enough to prove the theorem.

The proof of (8.5) and (8.6) is similar to the proof of parts (a) and (b) of the present theorem. We leave it to the reader. ∎

A problem connected with Theorem 20 is the determination of the points $(\lambda_r, \lambda_l) \in l_2 \backslash l_1$ such that $\alpha_1 > 0$ and $\alpha_2 = 0$ and those such that $\alpha_1 = 0$ and $\alpha_2 > 0$. Unfortunately we give only a partial answer.

Theorem 22. Suppose that $(\lambda_r, \lambda_l) \in l_2 \backslash l_1$. Then

(a) $\arctan(\lambda_l/\lambda_r) < \theta_c \Rightarrow \alpha_1(\lambda_r, \lambda_l) > 0$ and $\alpha_2(\lambda_r, \lambda_l) = 0$.

(b) $\arctan(\lambda_l/\lambda_r) > \pi/2 - \theta_c \Rightarrow \alpha_1(\lambda_r, \lambda_l) = 0$ and $\alpha_2(\lambda_r, \lambda_l) > 0$

Proof. (b) follows from (a) by symmetry. To prove (a), we use Theorems 4 and 5 to write for $\lambda_r > 4$, $\lambda_l > 1$

$$\alpha_1(\lambda_r, \lambda_l) \geqslant \alpha_1(4, \lambda_l) + \lambda_r - 4 \geqslant \alpha_1(4, 1) + \lambda_r - 4$$

From Proposition 3b it follows that $\bar{\lambda}_r(\lambda_l)$ is as large as we want if λ_l is sufficiently close to 1. Since $\alpha_1(4, 1) > -\infty$, there exists $a > 1$ such that

$$(\lambda_r, \lambda_l) \in l_2,\ \lambda_l \leqslant a \Rightarrow \alpha_1(\lambda_r, \lambda_l) > 0,\ \alpha_2(\lambda_r, \lambda_l) = 0$$

But by Theorems 15 and 9, $\theta \to \bar{\alpha}_i(\lambda_{c2}(\theta), \theta) = \alpha_i(\lambda_{c2}(\theta)\, c(\theta), \lambda_{c2}(\theta)\, s(\theta))$ $(i = 1, 2)$ are continuous functions. If there were a point $(\lambda_r, \lambda_l) \in l_2$ such that $\alpha_2(\lambda_r, \lambda_l) > 0$ and $\alpha_1(\lambda_r, \lambda_l) = 0$ and $\arctan(a/\bar{\lambda}_r(a)) < \arctan(\lambda_l/\lambda_r) < \theta_c$, then there would be another point $(\tilde{\lambda}_r, \tilde{\lambda}_l)$ such that $\arctan\ (\tilde{\lambda}_l/\tilde{\lambda}_r) < \theta_c$ and $\alpha_1(\tilde{\lambda}_r, \tilde{\lambda}_l) = \alpha_2(\tilde{\lambda}_r, \tilde{\lambda}_l) = 0$, which contradicts the definition of θ_c. ∎

9. SOME OPEN PROBLEMS

(1) One obvious important problem is the behavior of the process on l_1. But this is just an extension of the same still open problem in the symmetric case.

(2) Is $\theta_c = \pi/4$?

(3) If the answer to (2) is negative, are there values $\theta_c < \theta < \pi/4$ such that $\lambda_{c1}(\theta) < \lambda_{c2}(\theta)$?

(4) Get more information about the geometry of l_1 and l_2, for instance the concavities.

ACKNOWLEDGMENT

We thank Joel Lebowitz for his warm hospitality at Rutgers, where this work was finished.

When the first version of this paper was finished, we learned that A. Galves and E. Prescutti proved independently a statement similar to Theorem 20 in the case of the BCP when the system is seen from a moving frame of reference.

This paper benefited from the constructive criticism of a referee.

This work was partially supported by CNP_q (Brazil) and NSF Grant No. DMR 81-14726.

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